Limits of functions

1 A function $f(x)$ has the *limit* L as x approaches a if, for every positive number ε, there exists a positive number δ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$. The limit of a function is denoted by

$$\lim_{x \to \infty} f(x) = L.$$

Next, we consider the properties of limits.

2 The *limit of a constant* is equal to the constant:

$$\lim_{x \to a} C = C$$

3 The *limit of the sum of two functions* is equal to the sum of their limits (assuming these limits exist – this remark applies also to other formulas below):

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

4 The *limit of the difference of two functions* is the difference of their limits:

$$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

5 The *limit of the product of two functions* is equal to the product of their limits:

$$\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

6 The *limit of the quotient of two functions* is the quotient of their limits, provided the limit in the denominator is non-zero:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \text{ if } \lim_{x \to a} g(x) \neq 0.$$

7 A constant factor can be taken out of the limit:

$$\lim_{x \to a} [kf(x)] = k \lim_{x \to a} f(x)$$

8 *Limit of a composite function*

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

9 *Limit of a continuous function*

If the function $f(x)$ is continuous at $x = a$, then

$$\lim_{x \to a} f(x) = f(a).$$

Some special limits involving trigonometric functions

10 $\lim_{x\to 0} \frac{\sin x}{x} = 1$

11 $\lim_{x\to 0} \frac{\tan x}{x} = 1$

12 $\lim_{x\to 0} \frac{\arcsin x}{x} = 1$

13 $\lim_{x\to 0} \frac{\arctan x}{x} = 1$

14 $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$

Some special limits involving exponential functions

15 $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$

16 $\lim_{x\to\infty} \left(1+\frac{k}{x}\right)^x = e^k$

17 $\lim_{x\to 0} a^x = 1$

Limits of functions – the indeterminate forms

Indeterminate Forms $\frac{0}{0}$

Let $f(x)$ and $g(x)$ be two functions such that

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0.$$

Then the function $\frac{f(x)}{g(x)}$ has the indeterminate form $\frac{0}{0}$ at $x = a$. To find the limit at $x = a$ when the function $\frac{f(x)}{g(x)}$ has the indeterminate form $\frac{0}{0}$ at this point, we must factor the numerator and denominator and then reduce the terms that approach zero.

Indeterminate Forms $\frac{\infty}{\infty}$

Let $f(x)$ and $g(x)$ be two functions such that

$$\lim_{x\to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to a} g(x) = \pm\infty.$$

where a where a is a real number, or $+\infty$ or $-\infty$. It is said that the function $\frac{f(x)}{g(x)}$ has the indeterminate form $\frac{\infty}{\infty}$ at this point. To find the limit, we must divide the numerator and denominator by x of highest degree.

Indeterminate Forms $\infty - \infty$, $0 \cdot \infty$, ∞^0, 1^∞

Indeterminate forms of these types can usually be treated by putting them into one of the forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Exercises

✓ Example 1

Find the limit $\lim_{x\to 1} \frac{x^{20}-1}{x^{10}-1}$.

✓ Example 2

Calculate $\lim_{y\to -2} \frac{y^3+3y^2+2y}{y^2-y-6}$.

✓ Example 3

Calculate $\lim_{x\to \infty} \frac{x^3+3x+5}{2x^3-6x+1}$.

✓ Example 4

Calculate the limit $\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{x-1}$.

✓ Example 5

Find the limit $\lim_{x\to \pi} \frac{\cos \frac{x}{2}}{\pi-x}$.

✓ Example 6

Calculate $\lim_{x\to\infty} \left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)$.

✓ Example 7

Find the limit $\lim_{x\to 4} \frac{\sqrt{1+6x}-5}{\sqrt{x}-2}$.

✓ Example 8

Find the limit $\lim_{x\to\infty} \frac{(2x+3)^{10}(3x-2)^{20}}{(x+5)^{30}}$.

✓ Example 9

Find the limit $\lim_{x\to e} \frac{\ln x-1}{x-e}$.

✓ Example 10

Find the limit

$\lim_{t\to +\infty} \left(\sqrt{t+\sqrt{t+1}}-\sqrt{t}\right)$.

The number e

The number e is defined by:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$$

The number e is a transcendental number which is approximately equal to $2.718281828\ldots$
The substitution $u = \frac{1}{n}$ where $u = \frac{1}{n} \to 0$ as $n \to \pm\infty$, leads to another definition for e :

$$e = \lim_{u \to 0} (1+u)^u.$$

Here we meet with power expressions, in which the base and power approach to a certain number a (or to infinity). In many cases such types of limits can be calculated by taking logarithm of the function.

Exercises

✓ Example 1

Calculate the limit $\lim\limits_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n+5}$.

✓ Example 2

Find the limit $\lim\limits_{x \to \infty} \left(1 + \frac{1}{x}\right)^{3x}$.

✓ Example 3

Calculate the limit $\lim\limits_{x \to \infty} \left(1 + \frac{6}{x}\right)^{x}$.

✓ Example 4

Find the limit $\lim\limits_{x \to 0} \sqrt[x]{1+3x}$.

✓ Example 5

Find the limit $\lim\limits_{x \to \infty} \left(\frac{x+a}{x-a}\right)^{x}$.

✓ Example 6

Calculate the limit $\lim\limits_{x \to \infty} \left(\frac{x}{x+1}\right)^{x}$.

✓ Example 7

Evaluate the limit $\lim\limits_{x \to \infty} \left(\frac{x+3}{x-2}\right)^{x-1}$.

✓ Example 8

Find the limit $\lim\limits_{x \to a} \frac{\ln x - \ln a}{x-a}$, $(a > 0)$.

✓ Example 9

Calculate the limit $\lim\limits_{x \to 0} (1 + \sin x)^{\frac{1}{x}}$.